Introduction to Coordinate Proofs

Proofs involving midpoints, slope, and distance can be simplified by using analytic geometry. These proofs are called **coordinate proofs**. In a coordinate proof, the figure is drawn and labeled on a coordinate plane in a way that makes finding distances easy. Begin by placing one vertex of the figure at the origin. Place one side of the figure on the x-axis. Place parallel lines on either horizontal or vertical lines. Use a horizontal line and a vertical line for perpendicular lines. Once the figure has been placed on the coordinate plane, the distance formula can be used to measure distances, the midpoint formula can be used to locate points, and the slope formula can be used to determine parallel or perpendicular lines. Coordinate proofs rely on the premises of geometry plus the following properties from algebra.

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### Coordinate Geometry Formulas

The distance between points \((x_1, y_1)\) and \((x_2, y_2)\) is \(\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}\).

The midpoint of the segment joining the points \((x_1, y_1)\) and \((x_2, y_2)\) is \(\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)\).

The slope \(m\) of a line through two points \((x_1, y_1)\) and \((x_2, y_2)\), \(x_1 \neq x_2\), is \(m = \frac{y_2 - y_1}{x_2 - x_1}\).

The slope of a horizontal line is zero.

The slope of a vertical line is undefined.

Two lines with slopes \(m_1\) and \(m_2\) are parallel if and only if \(m_1 = m_2\).

Any vertical line is perpendicular to any horizontal line.

Two non-vertical lines are perpendicular if and only if their slopes are negative reciprocals of each other.

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1. Write a coordinate proof of the conjecture:

   *The diagonals of a square are congruent and are perpendicular bisectors of each other.*

   a. For the conditional, rewrite the statement in if-then form, and then label the hypothesis and the conclusion.

   b. Place and label the figure on the coordinate plane.

      (1) Place one vertex, point A, at the origin.

      ![Diagram](A(0, 0))
Introduction to Coordinate Proofs (Continued)

(2) Place a second vertex, point B, on the x-axis. This simplifies calculations because the y-coordinate of this point is 0.

\[ \text{Diagram of} \ A(0,0) \quad \text{and} \quad B(a,0) \]

(3) \( \overline{AD} \) and \( \overline{AB} \) must be perpendicular. Since \( \overline{AB} \) lies on the x-axis, \( \overline{AD} \) must lie on the y-axis a units above point A.

\[ \text{Diagram of} \ A(0,0) \quad \text{and} \quad B(a,0) \]

(4) Place point C a units above point B.

\[ \text{Diagram of} \ A(0,0) \quad \text{and} \quad B(a,0) \]

(5) Draw the diagonals.

\[ \text{Diagram of} \ A(0,0) \quad \text{and} \quad B(a,0) \]
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b. Write the given and prove of the conditional statement in terms of the diagram.

Given: Square ABCD with diagonals \(AC\) and \(BD\)
Prove: \(AC \perp BD\)
\(AC \cong BD\)
\(AC\) and \(BD\) bisect each other

c. Use the distance formula to find the lengths of the two diagonals.

\[
AC = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2} = \sqrt{(a-0)^2 + (a-0)^2} = \sqrt{2a^2} = a\sqrt{2}
\]
\[
BD = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2} = \sqrt{(a-0)^2 + (0-a)^2} = \sqrt{2a^2} = a\sqrt{2}
\]

So, by the definition of congruence, \(AC \cong BD\) because they both have the same lengths.

c. Use the midpoint formula to find the midpoints of the two diagonals.

Midpoint of \(AC\) = \(\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)\) = \(\left(\frac{0 + a}{2}, \frac{0 + a}{2}\right)\) = \(\left(\frac{a}{2}, \frac{a}{2}\right)\)
Midpoint of \(BD\) = \(\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)\) = \(\left(\frac{0 + a}{2}, \frac{a + 0}{2}\right)\) = \(\left(\frac{a}{2}, \frac{a}{2}\right)\)

So, \(AC\) and \(BD\) bisect each other because both segments have the same midpoint.

d. Use the slope formula to compare the slopes of the two diagonals.

Slope of \(AC\) = \(\frac{y_2 - y_1}{x_2 - x_1}\) = \(\frac{a - 0}{a - 0}\) = 1
Slope of \(BD\) = \(\frac{y_2 - y_1}{x_2 - x_1}\) = \(\frac{a - 0}{0 - a}\) = -1

So, \(AC \perp BD\) because the product of the slopes of the two segments is \(-1\).

Therefore, the diagonals of a square are congruent and are perpendicular bisectors of each other.

2. Write a coordinate proof of the conditional statement:

The segment that joins the midpoints of two sides of a triangle
(1) is parallel to the third side of the triangle, and
(2) has a length equal to half the length of the third side.

a. For the conditional, rewrite the statement in if-then form, and then label the hypothesis and the conclusion.
b. Draw and label a figure on the coordinate plane.  
*Hint: The algebra to calculate the coordinates of the midpoints of the two sides of the triangle can be simplified if you multiply each of the coordinates of the vertices of the triangle by two.*

![Diagram of a triangle with midpoints](image)

Given: Triangle ABC  
M is the midpoint of AB  
N is the midpoint of AC

Prove:  
1. $\overline{MN} \parallel \overline{BC}$  
2. $MN = \frac{BC}{2}$

c. Write the given and prove in terms of the diagram.

Given: Triangle ABC  
M is the midpoint of AB  
N is the midpoint of AC

Prove:  
1. $\overline{MN} \parallel \overline{BC}$  
2. $MN = \frac{BC}{2}$

d. Use the midpoint formula to find the coordinates of the midpoints of the two congruent sides of the triangle.

\[
M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) = \left(\frac{2b + 0}{2}, \frac{2c + 0}{2}\right) = (b, c)
\]

\[
N = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) = \left(\frac{2a + 2b}{2}, \frac{2c + 0}{2}\right) = (a + b, c)
\]

e. Use the slope formula to compare the slopes of the two segments.

Slope of $\overline{MN} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{c - c}{a + b - b} = \frac{0}{a} = 0$

Slope of $\overline{BC} = \frac{0-0}{2a-0} = \frac{0}{a} = 0$

So, $\overline{MN} \parallel \overline{BC}$ because the two segments have equal slopes.
Introduction to Coordinate Proofs (Continued)

f. Use the distance formula to find the lengths of the two segments.

\[ MN = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(a + b - b)^2 + (0 - 0)^2} = \sqrt{(a)^2 + (0)^2} = a \]
\[ BC = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(2a - 0)^2 + (0 - 0)^2} = \sqrt{(2a)^2 + (0)^2} = 2a \]

So, \( MN = \frac{BC}{2} \) because \( a = \frac{1}{2}(2a) \).

Therefore, the segment that joins the midpoints of two sides of a triangle is (1) parallel to the third side of the triangle, and (2) equal in length to one-half the length of the third side.
Answers:  
1. a. If a quadrilateral is a square, then the diagonals are congruent and are perpendicular bisectors of one another.

2. a. If a segment joins the midpoints of two sides of a triangle, then the segment is parallel to the third side of the triangle and has a length that is one-half of the length of the third side.